V Zero - points.

1. Def.

if $f(z)$ defined in $\mathcal{D}$, and $f(a)=0$, $a \in \mathcal{D}$, then a is zero point.

if for $z \in \{z: |z-a| < R\}$ that $f(z)$ is not always 0, then the power series expanded at a must not be all zeros.

So, if $f(a) = f^{(1)}(a) = \cdots = f^{(m-1)}(a) = 0$, but $f^{(m)}(a) \neq 0$.

then a is the m-th order zero point.

2. Theorem

$f(z)$ has a m-th order zero point and it's not always 0

iff

$f(z) = (z-a)^m \varphi(z)$, $\varphi(z)$ is analytic in $|z-a| < R$, $\varphi(a) \neq 0$

This is a very useful theorem to check the order of zero point.

3. Zero points are isolated.

If for z s.t. $|z-a| < R$, $f(z)$ is not always. 0, a is one zero points. Then, there must be $\varepsilon > 0$ s.t. $N_\varepsilon(a)$ is a region only has one zero point a. (zero points are not side-by-side, they are isolated)

Proof: a is the m-th zero point of $f(z)$, and $f(z)$ is not all zero.

So, $f(z) = (z-a)^m \varphi(z)$ $\quad \varphi(a) \neq 0$.

Then $\because$ $\varphi(z)$ analytic, so $\exists$ a region $N_\varepsilon(a)$ that $\varphi(z) \neq 0$

which means $f(z) = \underbrace{(z-a)^m}_{\neq 0} \underbrace{\varphi(z)}_{\neq 0}$ when $z \neq a$.

Corollary

if $f(z)$ analytic in $|z-a|<R$ and exists $\{z_n\}$ zero-point sequences s.t. $\{z_n\} \xrightarrow{n \to \infty} a$

Then $f(z)$ must be all zeros in $|z-a|<R$

4. Uniqueness

If $f_1(z)$ & $f_2(z)$ analytic in D, and a seq $\{z_n\}$ in D and converges to a $(z_n \neq a)$ where $f_1(z_n) = f_2(z_n)$

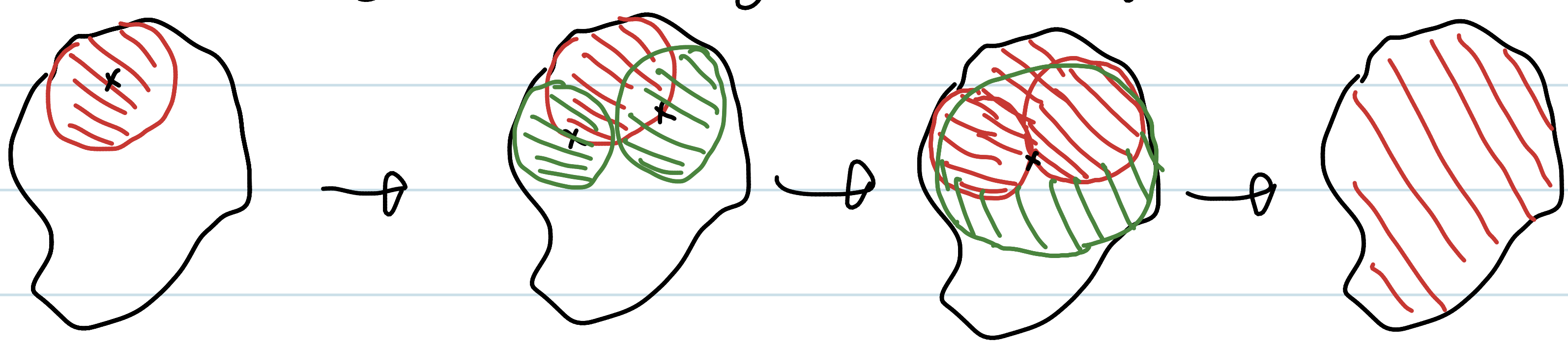
Then $f_1(z) = f_2(z)$ $\forall z$ in D.

Proof: let $f(z) = f_1(z) - f_2(z)$

$f(z)$ is analytic as $f_1(z)$ & $f_2(z)$ analytic.

If D is a circle centered at a, from the corollary above, we know $f(z)$ is always 0 in D

If D is not a circle, we can use a series of circles to represent this region. Sequentially, we can prove that "all zero" region becomes larger and larger, until equals to D.



5. Maximum Module Principle.

If $f(z)$ is analytic in $\mathcal{D}$, $|f(z)|$ can not reach maximum unless $|f(z)|$ is a constant.

Proof. From Mean Value Theorem (topic 3, page 4)

$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + Re^{i\varphi})\, d\varphi$ R is selected to make sure $a+Re^{i\varphi} \in \mathcal{D}$. so, it's an average.

If $|f(a)|$ reach $|f(z)|$'s maximum M, then

$$|f(a)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(a+Re^{i\varphi})\, d\varphi\right|$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi} |f(a+Re^{i\varphi})|\, d\varphi$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi} M\, d\varphi$$
$$= M$$

$\therefore |f(a+Re^{i\varphi})| = M$ for $\forall \varphi$ and R

$\Rightarrow |f(z)| = M$.